Chapter 1 Sequences

1.1 Sequences and Convergence

1. Show that \([0; 1]\) is a neighborhood of \(\frac{2}{3}\). That is, there is \(\epsilon > 0\) such that

\[
\frac{2}{3} - \epsilon < x < \frac{2}{3} + \epsilon \quad \text{for some } x \in [0; 1].
\]

Choose \(\epsilon = \frac{1}{3}\). Then \(\frac{2}{3} - \frac{1}{3} < \frac{2}{3} + \frac{1}{3} = \frac{4}{3} \in [0; 1]\).

2. Let \(x\) and \(y\) be distinct real numbers. Prove there is a neighborhood \(P\) of \(x\) and a neighborhood \(Q\) of \(y\) such that \(P \cap Q = \emptyset\).

Choose \(\epsilon = \min\{x - y, y - x\}\). Then, let \(P = (x \ - \epsilon; x + \epsilon)\) and \(Q = (y \ - \epsilon; y + \epsilon)\). Then, \(P \cap Q = \emptyset\).

3. Suppose \(x\) is a real number and \(\epsilon > 0\). Prove that \((x \ - \epsilon; x + \epsilon)\) is a neighborhood of each of its members; in other words, if \(y \in (x \ - \epsilon; x + \epsilon)\), then there is \(\delta > 0\) such that \((y \ - \delta; y + \delta) \in (x \ - \epsilon; x + \epsilon)\).

4. \( \sum_{n=1}^{\infty} \frac{1}{2n^2} \).

Find upper and lower bounds for the sequence \(\sum_{n=1}^{\infty} \frac{1}{2n^2} \).

First, \(n = 3 + n\). Thus, the lower bound is 3 and the upper bound is 10.
5. Give an example of a sequence that is bounded but not convergent.

Let $a_n = (1)^n$. Then, this sequence alternates between 1 and 1, but never converges.
6. Use the definition of convergence to prove that each of the following sequences converges:

(a) \(5 + \frac{1}{n}\)

(b) \(\frac{2^n}{n}\)

(c) \(\frac{3n}{2}\)

(d) \(\frac{2n+1}{n}\)

(a) Let \(\varepsilon > 0\) be given. Let \(N = 1\). Then, \(\frac{5}{\varepsilon} < n < \frac{5}{\varepsilon}\) for all \(n \geq N\).

(b) Let \(\varepsilon > 0\) be given. Let \(N = \frac{2^n}{\varepsilon}\). Then, \(\frac{2^n}{\varepsilon} < n < \frac{2^n}{\varepsilon}\) for all \(n \geq N\).

(c) Let \(\varepsilon > 0\) be given. Let \(N = \frac{3n}{\varepsilon}\). Then, \(\frac{3n}{\varepsilon} < n < \frac{3n}{\varepsilon}\) for all \(n \geq N\).

(d) Let \(\varepsilon > 0\) be given. Let \(N = \frac{2n+1}{\varepsilon}\). Then, \(\frac{2n+1}{\varepsilon} < n < \frac{2n+1}{\varepsilon}\) for all \(n \geq N\).
7. Show that \( f_{n}^{1} \) converges to \( A \) if \( f_{n} \) \( \to \) \( A \). \( g_{n}^{1} \) converges to 0.

We see that \( j(a_{n} - A) = \sum_{n=1}^{\infty} 0 = j(a_{n}) < . \)

8. Suppose \( f_{n}^{1} \) converges to \( A \), and define a new sequence \( f_{n}^{1} \) by \( b_{n} = \frac{a_{n}}{2} \) for all \( n \). Prove that \( f_{n}^{1} \) converges to \( A \).

Let \( \epsilon \) be given. We see that

\[ \sum_{n=1}^{\infty} 2 = \sum_{n=1}^{2 \epsilon} a_{n} \]

Thus, \( b_{n} \to A \).

9. Suppose \( f_{n}^{1} \), \( b_{n} \), \( g_{n}^{1} \), and \( b_{n} \) such that \( f_{n}^{1} \) converges to \( A \), \( f_{n}^{1} \) converges to \( A \), and \( a_{n} \) \( c_{n} \) \( b_{n} \) for all \( n \). Prove that \( f_{n}^{1} \) converges to \( A \).

Since \( a_{n} c_{n} b_{n} \), we must have \( a_{n} c_{n} A b_{n} \). By convergence and definition of absolute value, \( < a_{n} A c_{n} b_{n} A < . \) Hence, \( j c_{n} A j < . \) Thus, \( c_{n} \to A \). (We will call this result the Squeeze Theorem.)

10. Prove that, if \( f_{n}^{1} \) converges to \( A \), then \( f_{n} \) \( f_{n} \) converges to \( j A j \). Is the converse true? Justify your conclusion.

We see that \( j(1) j \to j(1) j = j(1) \to j \). The converse is not true. For instance, \( j(1) j \) \( 1 \), but \( f(1) j \) diverges.

11. Let \( f_{n}^{1} \) be a sequence such that there exist numbers \( N \) such that, for \( n N \), \( a_{n} = . \) Prove that \( f_{n}^{1} \) converges to \( . \)

We see that \( j a_{n} j \) \( 0 \) \( for all \( > 0 \).
12. Give an alternate proof of Theorem 1.1 along the following lines. Choose \( \varepsilon > 0 \). There is \( N_1 \) such that for \( n > N_1 \), \( |a_n - A| < \frac{\varepsilon}{2} \), and there is \( N_2 \) such that for \( n > N_2 \), \( |b_n - B| < \frac{\varepsilon}{2} \). Use the triangle inequality to show that this implies that \( |A - B| < \varepsilon \).

Let \( N = \max(N_1, N_2) \).

Thus, \( |A - B| < \varepsilon \).\( \square \)

13. Let \( x \) be any positive real number, and define a sequence \( \{a_n\} \) by
\[
 a_n = [x] + [2x] + \ldots + [nx]
\]
where \([x]\) is the largest integer less than or equal to \( x \). Prove that \( \{a_n\} \) converges to \( x = \frac{1}{2} \).

Let \( \varepsilon > 0 \) be given and set \( N = \frac{x}{\varepsilon} \). Then,
\[
|a_n - \frac{x}{2} - \frac{n\varepsilon}{2} - \frac{n\varepsilon}{2}| = |\varepsilon - \frac{n\varepsilon}{2}| < \varepsilon.
\]

1.2 Cauchy Sequences

14. Prove that every Cauchy sequence is bounded. (Theorem 1.4)

Suppose that \( \{a_n\} \) is not bounded. Then, for any \( k \), there is an \( n_k \) such that \( |a_{n_k} - A| > k \). Then, \( \{a_{n_k}\} \) is an unbounded sequence. Then, for any \( N \), there exist \( a_{n_k} \) and \( a_{n} \) such that \( |a_{n_k} - a_{n_k}| > |a_{n} - a_{n_k}| = k \) where \( k > N \). Thus, \( \{a_n\} \) is not Cauchy.

15. Prove directly (do not use Theorem 1.8) that, if \( \{a_n\} \) and \( \{b_n\} \) are Cauchy, so is \( \{a_n + b_n\} \).

Since \( \{a_n\} \) and \( \{b_n\} \) are Cauchy, then for all \( \varepsilon > 0 \), there exist \( N_1 \) and \( N_2 \) such that \( |a_{m} - a_{n}| < \frac{\varepsilon}{2} \) for all \( m, n > N_1 \) and \( |b_{m} - b_{n}| < \frac{\varepsilon}{2} \) for all \( m, n > N_2 \). Choose \( N = \max(N_1, N_2) \). Then, \( |a_{m} + b_{m} - a_{n} - b_{n}| = |a_{m} - a_{n}| + |b_{m} - b_{n}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \) for all \( m, n > N \).

16. Prove directly (do not use Theorem 1.9) that, if \( \{a_n\} \) and \( \{b_n\} \) are Cauchy, so is \( \{a_n b_n\} \). You will want to use Theorem 1.4.

Since \( \{b_n\} \) is Cauchy, then it is bounded (by Exercise 14). Thus, \( |b_{n}| < M \) for some \( M \). Since \( \{a_n\} \) is Cauchy, then for all \( \varepsilon > 0 \), there exist \( N \) such that \( |a_{n} - a_{m}| < \varepsilon M \) for all \( m, n > N \) and \( |b_{n} b_{m} - b_{n} b_{m}| < \varepsilon M \) for all \( m, n > N \). Let \( \varepsilon > 0 \) be given. Then, \( |a_{n} b_{m} - a_{n} M| < \varepsilon M |a_{m} - a_{n}| \) for all \( m, n > N \).
17. Prove that the sequence $\frac{2n+1}{n}$ is Cauchy.

Let $\varepsilon > 0$ be given. Choose $N = \frac{1}{\varepsilon}$. Then,

$$\frac{2m+1}{n} - \frac{2n+1}{m} = \frac{2nm + n - 2mn - m}{nm} = \frac{n - m}{nm} < \varepsilon$$

18. Give an example of a sequence with exactly two accumulation points.

Let $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 + \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$. Then, $a_n$ has accumulation points at 0 and 1.

19. Give an example of a set with a countably infinite set of accumulation points.

The set $\mathbb{Q}$ has the property that every element is an accumulation point, since for any $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, the sequence $\frac{a}{b} + \frac{1}{n}$ converges to $\frac{a}{b}$. Since $\mathbb{Q}$ is countable, we have found the desired set.

20. Give an example of a set that contains each of its accumulation points.

The set $[0; 1]$ contains all of its accumulation points.

21. Determine the accumulation points of the set $2^n + \frac{k}{n}$.

The set $\{2^n + \frac{k}{n} : n \in \mathbb{Z}^+\}$ is the set of accumulation points since $2^n + \frac{k}{n} \to 2^n$ as $k \to 0$ and $2^n + \frac{k}{n} \to 1$ as $n \to 1$.

22. Let $S$ be a nonempty set of real numbers that is bounded from above (below) and let $x = \sup S$ (inf $S$). Prove that either $x$ belongs to $S$ or $x$ is an accumulation point of $S$.

It is clear that $x \in S$ is a possibility. Suppose $x \not\in S$. Then, by Exercise 0.44, for any $\varepsilon > 0$, there is an $a \in S$ such that $x < a < x$. Thus, for all $n$, there exists an $a_n \in S$ such that $x - \frac{1}{n} < a_n < x$. Since $x - \frac{1}{n} \not\to x$, we have $a_n \not\to x$. Thus, $x$ is an accumulation point of $S$.

23. Let $a$ and $1 > a$ be distinct real numbers. Define $a_n = a_{n-1} + 1$ for each positive integer $n > 2$. Show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. You may want to use induction to show that

$$a_{n+1} - a_n = 2(a_{1} - a_{0})$$

and then use the result from Example 0.9 of Chapter 0.
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The statement \( a_{n+1} = 2^n (a_1 - a_0) \) is obviously true for \( n = 1 \). Suppose that it is true for \( n < N \). Then, \( a_{N+2} = \frac{a_{N+1} + 2^n a_{N+1}}{2^{n+1}} \). Suppose \( a_{N+1} = \frac{a_{N+1} + 2^n a_{N+1}}{2^{n+1}} \) and \( a_{N} = \frac{a_{N} + 2^n a_{N} + 1}{2^{n+1}} \). Then, \( a_{N+2} = \frac{a_{N+1} + 2^n a_{N+1}}{2^{n+1}} \).

Now, let \( N > 0 \) and \( n; m \) be given. Choose \( N = \lg \left( \frac{a_n a_m}{a_n a_m + 1} \right) \).

Then, \( \frac{a_n + a_m}{2^n} \) converges to \( A \) for \( n \) if and only if \( \frac{a_n + a_m + 1}{2^n} \) converges to \( A \). Thus, we see that \( a_n a_m \) converges to \( A \).

Now, every neighborhood of \( A \) has \( N_k \) as a subset for some \( k \). Since there are \( \infty \) \( N_k \)'s, we have \( \infty \) members of \( f_{n} a_n \) in any neighborhood.

1.3 Arithmetic Operations on Sequences

24. Suppose \( f_{n} a_n \) converges to \( A \) and \( f_{n} a_n \) is an ininite set.

Show that \( A \) is an accumulation point of \( f_{n} a_n \) for \( n \geq 2 \). By Theorem 1.8, \( f_{n} a_n \) converges to \( A \).

25. Suppose \( f_{n} a_n \) and \( f_{n} b_n \) are sequences such that \( f_{n} a_n \) and \( f_{n} b_n \) converge. Prove that \( f_{n} a_n \) converges to \( A \).

Suppose \( a_n \) converges to \( A \) and \( a_n + b_n \) converges to \( C \). Then, \( f_{n} a_n + f_{n} b_n \) converges. Thus, by Theorem 1.8, \( a_n + b_n \) converges.

26. Give an example in which \( f_{n} a_n \) and \( f_{n} b_n \) do not converge but \( f_{n} a_n + f_{n} b_n \) converges.

Let \( a_n = (1)^n \) and \( b_n = (1)^{n+1} \). We know that \( a_n \) and \( b_n \) don't converge, but \( a_n + b_n \) converges.

27. Suppose \( f_{n} a_n \) and \( f_{n} b_n \) are sequences such that \( f_{n} a_n \) converges to \( A \). Then, \( f_{n} b_n \) converges. Prove that \( f_{n} b_n \) converges.

Suppose \( a_n b_n \) converges to \( C \). Then, \( f_{n} b_n \) converges. Thus, by Theorem 1.9, \( a_n \) converges.

28. If \( a \) converges to \( A \) with \( a_n \) for all \( n \), show that \( f_{n} a_n \) converges to \( A \).
Let $\varepsilon > 0$ be given. Then, there is an $N$ such that $a_n < \varepsilon$. Then,

\[a_n < \varepsilon\]

for all $n \geq N$.

29. Prove that

\[
\frac{n + k}{k} > (n + k)^{k^n}
\]

converges to $\frac{1}{e}$, where
\[
\frac{n + k}{k} = \frac{(n + k)!}{n!k!}.
\]

We see that \(\frac{(n + k)!}{n!k!} = \frac{1}{(n + k)}\cdot \frac{1}{(n + k - 1)}\cdot \ldots \cdot \frac{1}{1}\). Now, let \(\epsilon > 0\) be given. Choose \(N = \frac{1}{\epsilon}\). Then, for all \(n > N\),

\[
\frac{n + k}{k} = \frac{n + k}{n!} < \epsilon.
\]

30. Prove the following variation on Lemma 1.10. If \(f_{n} g_{n}\) \(n \geq 1\) converges to \(B \neq 0\) and \(b_{n} \neq 0\) for all \(n\), then there is \(M > 0\) such that \(|b_{n}| \geq M\) for all \(n\).

Choose \(M = |b_{n}|\). Then, the statement holds.

31. Consider a sequence \(f_{n} g_{n}\) \(n \geq 1\) and, for each \(n\), define

\[
\alpha_{n} = \frac{a_{1} + a_{2} + \ldots + a_{n}}{n}.
\]

Prove that if \(f_{n} g_{n}\) \(n \geq 1\) converges to \(A\), then \(f_{n} g_{n}\) \(n \geq 1\) converges to \(A\). Give an example in which \(f_{n} g_{n}\) \(n \geq 1\) converges, but \(f_{n} g_{n}\) \(n \geq 1\) does not.

Let \(\epsilon > 0\) be given. There is an \(N_{1}\) such that \(|a_{n} - A| < \epsilon\) for all \(n > N_{1}\). Let \(M = |a_{1} + a_{2} + \ldots + a_{N_{1}} - A|\). Then, there is an \(N_{2}\) such that \(M < \epsilon\) for all \(n > N_{1}\). Let \(N = \max(N_{1}, N_{2})\). Then, for all \(n > N\),

\[
\frac{a_{1} + a_{2} + \ldots + a_{n}}{n} - A = \frac{a_{1} + a_{2} + \ldots + a_{N} - nA}{n} + \frac{a_{N+1} + a_{N+2} + \ldots + a_{n}}{n} < \epsilon.
\]

Let \(\alpha = (1 + 1)^{n}\). Then, as we have seen before, \(n\) \(n\) diverges, but \(\frac{a_{1} + a_{2} + \ldots + a_{n}}{n} \) diverges.

32. Find the limit of the sequences with general term as given:

(a) \(\frac{n^{2} + 2n}{n^{2}}\)

(b) \(\frac{\cos \frac{n}{n}}{\sin \frac{n}{n}}\)

(c) \(\frac{p^{n}}{n^{n}}\)
(d) \[ \frac{n}{n^2 - 3} \]

(e) \[ 4 \frac{n}{n^3 - 2} n \]

(f) \[ (1)^{n+7} \]

(a) 1

(b) 0

(c) 0

(d) 0

(e) \[ 4 \frac{1}{n-2} n = 4 \frac{1}{n-4n} \]

Thus, limit is \( 4 \).

(f) 0

33. Find the limit of the sequence in Exercise 23 when \( a_0 = 0 \) and \( a_1 = 3 \). You might want to look at Example 0.10.

34. Find a convergent subsequence of the sequence

\[ 1 \]

\[ \frac{1}{n} \]

Let \( n_k = 2n \). Then, the subsequence is \( \frac{2}{2} n \) which converges to 0.

35. Suppose \( x \) is an accumulation point of \( \{a_n : n \geq N \} \). Show that there is a subsequence of \( \{a_n : n \geq 1 \} \) that converges to \( x \).

Since \( x \) is an accumulation point, every neighborhood about \( x \) contains an in nity of \( a_n \). Thus, let \( a_{nk} \) be a member of \( \{a_n : n \geq 2 \} \) such that \( k \frac{1}{n} < x + k \frac{1}{n} \). Then, for any \( \epsilon > 0 \), there is a \( K \) such that \( k \frac{1}{n} < \epsilon \) for all \( k > K \). Thus, \( a_{nk} \to x \).

36. Let \( \{a_n : n \geq 1 \} \) be a bounded sequence of real numbers. Prove that \( \{a_n : n \geq 1 \} \) has a convergent subsequence.

Either \( \{a_n \} \) has a nite number of values or \( \{a_n \} \) has an in nite number of values. For the former, there must be some value \( x \) for which there are in nitely many \( k \) such that \( a_{nk} = x \). Thus, \( a_{nk} \to x \). For the latter, the sequence is a
bounded in finite set of real numbers, so by the Bolzano-Weierstrass Theorem, \( a_n \) has a convergent subsequence.

*37. Prove that if \( f_{a_n} \) is decreasing and bounded, then \( f_{a_n} \) converges.

Assume that \( f_{a_n} \) attains an infinite number of values. Suppose that \( \inf a_n = \infty \). Let \( \varepsilon > 0 \) be given. Then, there are \( \Delta \) intervals that sequence values may fall. Since this is a finite number and there are an infinite number of values, at least one region must contain an infinite number of function values. Since the sequence is decreasing, the last region must contain an infinite number of values; that is, \( a_n (M, M + \varepsilon) \) for all \( n > N \) for some \( N \). Since \( M \) was arbitrarily chosen, the proof is complete. The case for when \( f_{a_n} \) has only finitely many values is easy.

38. Prove that if \( c > 1 \), then \( \sqrt[n]{c^n} \) converges to 1.

Also, \( c > 1 \), so by Theorem 1.16, \( \sqrt[n]{c^n} \) converges to \( c \).

*39. Suppose \( f_{z_n} \) converges to \( x \) and \( f_{y_n} \) to \( y \). Define a sequence \( f_{z_n} \) as follows: \( z_{2n} = x_n \) and \( z_{2n + 1} = y_n \). Prove that \( f_{z_n} \) converges to \( x \).

Both subsequences of \( f_{z_n} \) converge to \( x \). Thus, by Theorem 1.14, \( z_n \) converges to \( x \).

40. Show that the sequence defined by \( a_1 = 6 \) and \( a_n = 6 + a_{n-1} \) for \( n > 1 \) is convergent and find its limit.

To find the limit \( L \), set \( L = \sqrt[6]{6 + L} \). Then, \( L^2 = 6 + L \). The solutions are 2 and 3. The only solution that works is 3. Thus, \( a \) is convergent. We prove that \( f_{a_n} \) is decreasing. Since \( 6 + a_{n-1} < 6 + a_{n-1} \) and we know that \( a_n \) is decreasing, we see that the whole sequence is decreasing. Also, square roots must be greater than 0, so the sequence is bounded. Thus, the sequence is bounded below and decreasing and is thus convergent.

41. Let \( f_{z_n} \) be a bounded sequence and let \( E \) be the set of subsequential limits of that sequence. By Exercise 36, \( E \) is nonempty. Prove that \( E \) is bounded and contains both sup \( E \) and inf \( E \).

Since \( f_{z_n} \) is bounded (by \( M \)), its limit points must be such that they are within distance of some sequence values. Thus, limit points must be within the same bounds as \( f_{z_n} \) or within distance of the boundary for any \( \varepsilon \). Thus, \( E \) is bounded (by, say \( M + 1 \)). We must ensure that members of \( E \) do not form a sequence themselves that converges to a non-limit point. So, suppose there is a sequence \( f_{z_n} \) of limit points. Then, for every \( \varepsilon \), there is an \( N \) such that
je_{nx} < for all \( n > N \). Thus, \( x_n \neq e_n \). Thus, all sequences of \( E \) converge in \( E \) (since they are estimated by subsequences of \( f \)). Thus, \( \sup E; \inf E \in E \).

42. Let \( f \) be any sequence and \( T : N \to N \) be any 1-1 function. Prove that if \( f \) converges to \( x \), then \( f(T(n)) \) also converges to \( x \). Explain how this relates to subsequences. Define what one might call a "rearrangement" of a sequence. What does the result imply about rearrangements of sequences?

We see that \( f(T(n)) = f(x_{n_1}; x_{n_2}; \ldots) \) and is a subsequence of \( f \). Since all subsequences converge, we must have \( x(T(n)) \neq x \). Let \( T : N \to N \) be any 1-1 function and let \( f \) be a sequence. Then, \( f(T(n)) \) is called a rearrangement. The result implies that if \( f \) converges, then so does \( f(T(n)) \) for any \( T \).

43. Assume \( 0 < a < b \). Does the sequence \( f(a^n + b^n) \) converge or diverge? If the sequence converges, find the limit.

The sequence does not converge in general. For instance, if \( a = 1 \) and \( b = 1 \), then the sequence becomes \( f(1^n + (1)^n) = 3^n \). Taking even indexes, the limit is 1, and taking odd indexes, the limit is 0. Thus, not all subsequences converge to the same limit point, so the sequence is not convergent.

44. Does the sequence

\[
\frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^n}}
\]

diverge or converge? If the sequence converges, find the limit.

We see that \( \frac{1}{k^n} < \frac{1}{k^{n-1}} \). Also, \( \frac{1}{k^n} > \frac{1}{k^{n+1}} \). Thus, the sequence must converge to 1 by the Squeeze Theorem (established in Exercise 9).

*45. Show that if \( x \) is any real number, there is a sequence of rational numbers converging to \( x \).

Let \( a_1a_2a_3 \ldots \) be the decimal expansion for \( x \). Then, define \( x_n = a_1a_2a_3 \ldots a_n \). Then, \( x_n \in Q \) for all \( n \) and \( x_n \neq x \) (for there exists an \( N \) for which \( x_n \) can be within \( 10^k \) distance for any integer \( k \) and \( n > N \)).

*46. Show that if \( x \) is any real number, there is a sequence of irrational numbers converging to \( x \).
If $x$ is already irrational, define $x_n \neq x$. Clearly, $x_n \neq x$. If $x$ is rational, define $x_n = x + n$. Then, $x_n \notin \mathbb{Q}$ for all $n$ and $x_n \neq x$. 

47. Suppose that \( f_n^{1 \rightarrow 1} \) converges to \( A \) and that \( B \) is an accumulation point of \( f_n: n \rightarrow N \). Prove that \( A = B \).

If \( B \) is an accumulation point, then \( B^{n \rightarrow 1} ; B + n^{1} \) contains a member of \( f_n \) for all \( n \). Thus, one can construct a subsequence of these members that converges to \( B \), and since \( f_n \) is convergent, we must have \( A = B \).

**Miscellaneous**

48. Suppose that \( f_n^{1 \rightarrow 1} \) and \( f_n^{1 \rightarrow 1} \) are two sequences of positive real numbers. We say that \( a_n \) is \( O(b_n) \) (read as \( \big\text{big oh} \)) if there is an integer \( N \) and a real number \( M \) such that for \( n \geq N \), \( a_n \leq M b_n \). Prove that if \( f_n^{1 \rightarrow 1} \) converges to \( L \neq 0 \), then \( a_n \) is \( O(b_n) \) and \( b_n \) is \( O(a_n) \). What can you say if \( L = 0 \)? Illustrate with examples.

Since \( a_n = b_n \Leftrightarrow L \), we guess that there is an \( N \) such that \( a_n \leq (L + 1) b_n \) for all \( n \geq N \). We now prove this assertion. First, for any \( > 0 \), there is an \( N \) for which

\[
a_n \leq 2(L + ) \quad \text{for all} \quad n \geq N.
\]

Thus, for the same \( N \),

\[
a_n \leq (L + 1) \quad \text{and} \quad b_n \text{is bounded or} \quad b_n \rightarrow 1 \quad \text{and} \quad a_n \text{is bounded.}
\]

For instance, \( n^3 \rightarrow 0 \) and \( n^2 \rightarrow 1 \) for \( n \rightarrow \infty \). (This result is called the Limit Comparison Test.)